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Entropy of Coupled Map Lattices

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The entropy of coupled map lattices with respect to the group of space-time translations is considered. We use the notion of generalized Lyapunov spectra to prove the analogue of the Ruelle inequality and the Pesin formula.

KEY WORDS: Coupled map lattice; Pesin formula; Ruelle inequality; SBR measure.

1. INTRODUCTION

The behavior of finite-dimensional hyperbolic diffeomorphisms is one of the best-developed branches of dynamical systems theory. Therefore the natural question arises as to which features of this behavior persist in the infinite-dimensional setting.

The simplest example to begin with is coupled map lattices. Here the configuration space $\mathbf{X}$ is the product of the countable number of copies of finite-dimensional manifolds $\mathbf{X} = \prod_i X_i$. In this paper the index i runs over integers ("one-dimensional lattice"). The map Φ is a perturbation of the product of "uncoupled" diffeomorphisms $(\mathbf{f}(\mathbf{x}))_i = f(\mathbf{x}_i)$ due to an interaction J which is translation invariant and rapidly decreasing in space. The last assumption can be formalized mathematically in many different ways. The simplest possibility is to require that f is a hyperbolic map and J is so small that methods of stability theory can be applied. Under these conditions it was shown in refs. 1 and 5 that the basic results of the finite-dimensional theory such as the stable-manifold theorem, the construction of Markov partitions, and the existence of SBR measure remain valid. The purpose of this paper is to generalize the entropy formulas.

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Let us recall that the classical Pesin formula states that in the ergodic case the measure-theoretic entropy is equal to the sum of positive Lyapunov exponents. It is based on the Oseledeĉ theorem. So the first problem is to obtain the counterpart to this theorem. The simplest proof of Oseledeĉ theorem[6] proceeds as follows. Let $\lambda_1^{(m)}(x) \geqslant \lambda_2^{(m)}(x) \geqslant \cdots \geqslant \lambda_n^{(m)}(x)$ be the eigenvalues of $[(d\Phi^m(x))^* (d\Phi^m(x))]^{1/2}$. Consider $s_j^m(x) = \sum_{k=1}^j \ln \lambda_k^{(m)}(x)$. The existence of the limit $\lim_{m \to \infty}(1/m)\, s_j^{(m)}$ follows by the subadditive ergodic theorem since one can interpret $\exp s_j^{(m)}$ as the largest eigenvalue of $d\Phi^m$ acting on j-forms. This approach succeeds also in infinite dimensions if $d\Phi$ is a compact operator.[3,9] In our situation the last assumption is never valid because of the translation invariance. However, a natural generalization arises if we want to compute the entropy with respect to the group of the *space-time* translations, that is, the "measure-theoretic entropy of the time shift per degree of freedom" rather then just the measure-theoretic entropy of the time shift. In this case we should average by the number of degrees of freedom N and perform the limit as N tend to infinity before applying the traditional arguments. This program was partly carried out in ref. 12 for another dynamical system: the hard-core gas in an infinite vessel. The above-mentioned approach to the Oseledeĉ theorem seems to be more natural than doing time averaging before the space averaging because while our system can be considered as a small perturbation of a finite-dimensional one for any *fixed* moment of time, the time average limit depends essentially on the whole infinite-dimensional space.

The important difference from the finite-dimensional case is that the generalized Lyapunov spectrum so obtained does not correspond to any invariant splitting of the tangent bundle, not to mention foliations. So its dynamical importance is not clear. However, in this paper we show that the counterparts to both the Ruelle inequality and the Pesin formula hold if the ordinary Lyapunov spectrum is replaced by the generalized one.

The structure of the paper is the following. Section 2 contains the precise assumptions about the interaction J. In Section 3 we define the expansion rate, which is the mean value of the sum of positive Lyapunov exponents. The existence of the limiting quantity is demonstrated in Section 4. Our arguments here are similar to those of ref. 11. After the existence is established, the Ruelle inequality follows by exactly the same arguments as in Ruelle's original paper.[7] This is discussed in Section 5. In Section 6 we recall the construction of the SBR measure for coupled map lattices given in ref. 5. The Pesin formula is proven in Section 7. The reason why it holds is that in the hyperbolic case the convergence in the Pesin formula for finite-dimensional system whose limit is our coupled map lattice is uniform in the number of degrees of freedom. The proof of the Pesin formula gives an affirmative answer to a general question posed in ref. 11 for our very special case.

Since it is interesting to find the weakest possible conditions under which this theory holds, we do not impose any hyperbolicity conditions in Sections 2–5. Of course this paper is only the first step toward understanding the entropy properties of differential infinite-dimensional systems.

2. COUPLED MAP LATTICES

Here we define the system with which we deal.[5] Let X be a compact Riemann manifold. Choose a countable number of copies X_i of X and set $\mathbf{X} = \prod_{i=-\infty}^{+\infty} X_i$, $X_{N_1,N_2} = \prod_{i=N_1}^{N_2} X_i$, $X_N = X_{-N,N}$. Elements of $\mathbf{X}$ are denoted by $\mathbf{x} = \{\mathbf{x}_i\}_{i=-\infty}^{+\infty}$ and elements of X_N by $x^{(N)}$. We write S for the space shift $(S(\mathbf{x}))_i = \mathbf{x}_{i+l}$. Denote by p_{N_1,N_2}, p_N and Q_N the natural projections $p_{N_1,N_2}\colon \mathbf{X} \to X_{N_1,N_2}$, $p_N\colon \mathbf{X} \to X_N$ and $Q_N\colon X_N \to X_{N-1}$. The distances on $\mathbf{X}$ and X_{N_l,N_2} are given by $d(x, y) = \sup_i \rho(x_i, y_i)$, where ρ is the distance on X. We write $V_i(x) = T_{x_i}(X_i)$. The tangent space $V(\mathbf{x}) = T_{\mathbf{x}}\mathbf{X}$ may be identified with $\bigoplus_i V_i(\mathbf{x}_i)$, with $\|v\| = \sup_i \|v_i\|$. We set

$$V^{N_1,N_2}(\mathbf{x}) = \bigoplus_{i=N_1}^{N_2} V_i(\mathbf{x}), \qquad V^N(\mathbf{x}) = V^{0,N-1}$$

(note, however, that $X_N = X_{-N,N}$) and P_{N_1,N_2} and P_N are the corresponding projections. We also consider the space $H(\mathbf{x})$ of vectors with a finite l_2-norm $\|v\|_2 = (\sum_i \|v\|^2)^{1/2}$.

Now we define our map. Let f be a diffeomorphism of X and $\mathbf{f}$ be the diffeomorphism of $\mathbf{X}$ given by $(\mathbf{f}(\mathbf{x}))_i = f(\mathbf{x}_i)$. We study diffeomorphisms of the form $\Phi = J \circ \mathbf{f}$, where J is an interaction map defined below. Let J_0 be a map $J_0\colon \mathbf{X} \to X$ such that there exist constants K_1 and $\kappa_1 < 1$ and mappings $J_0^{(N)}\colon X_N \to X$ such that

$$d_{C^2}(J_0^{(N)}, J_0^{(N-1)} Q_N) \leqslant K_1 \kappa_1^N \tag{1}$$

$$d_{C^2}(J_0, J_0^{(N)} p_N) \leqslant K_1 \kappa_1^N \tag{2}$$

A (K_1, κ_1) interaction is given by $(J(\mathbf{x}))_i = S^i J_0 S^{-i}(\mathbf{x})$. Since J is S-invariant, $\Phi S = S\Phi$. More general interactions can be considered as long as they satisfy conditions (3), (4) below. Let $D_n^m(\mathbf{x})$ be the diágonal part of $d\Phi^m$, that is, $D_n^m v = P_{i-n,i+n}\, d\Phi^m v$ if $v \in V_i(\mathbf{x})$. Clearly (1) and (2) imply

$$d\Phi\colon H(\mathbf{x}) \to H(\Phi\mathbf{x}) \qquad \text{and} \qquad \|d\Phi\|_2 \leqslant K_2 \tag{3}$$

[where $K_2 = K_1/(1-\kappa_1)$] and given ε, m there exists $n_0 = n_0(m)$ such that for all $n \geqslant n_0$

$$\|D_n'''(\mathbf{x}) - d\Phi'''(\mathbf{x})\|_2 \leqslant \varepsilon \tag{4}$$

Conditions (3) and (4) guarantee the existence of the expansion rate proven in Section 4.

3. EXPANSION RATE

In order to define the expansion rate we need some extra notations. If E and F are Hilbert spaces and $A: E \to F$ is a linear operator, we set $|A| = \sqrt{A^*A}$. In case E is finite-dimensional we denote by $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A)$ the eigenvalues of $|A|: E \to E$ and $\det A$ is the determinant of $|A|$. We call $\nu(A)$ the normalized counting measure of the eigenvalues $\nu(A) = (1/\dim E) \sum_j \delta_{\lambda_j(A)}$. The expansion rate of A can be defined as follows:

$$R(A) = \sum_{\lambda_j(A) > 1} \ln \lambda_j(A) = \dim E \int \ln^+(t)\, d\nu(A)(t)$$

where $\ln^+ t = \max(0, \ln t)$. Usually we consider the restriction of A to some finite-dimensional subspace $\bar{E} \subset E$. To avoid long subscripts we write $\lambda_j(A \mid \bar{E})$, $\det(A \mid \bar{E})$,... instead of $\lambda_j(A|_{\bar{E}})$, $\det(A|_{\bar{E}})$,.... For example,

$$R(A) = \max_{\bar{E} \subset E} \ln \det(A \mid \bar{E}) \tag{5}$$

Now we collect for future use some elementary properties of $\nu(A)$ and $R(A)$. The proofs are based on the observation that the inequality

$$\nu(A)([t, \infty]) \geqslant \frac{N}{\dim E} \qquad \left\{\nu(A)([0, t]) \geqslant \frac{N}{\dim E}\right\}$$

is equivalent to the existence of the subspace $\bar{E}$ of the dimension N on which $(Ae, Ae) \geqslant t^2(e, e)$ [$(Ae, Ae) \leqslant t^2(e, e)$ respectively].

Proposition 1. Let $\|A\| \leqslant a$; then:

(1) If $\|B\| \leqslant \varepsilon < a$, then

$$\nu(A)([\sqrt{t_1 + 3\varepsilon a}, \sqrt{t_2 - 3\varepsilon a}]) \leqslant \nu(A+B)([t_1, t_2]) \leqslant \nu(A)([\sqrt{t_1 - 3\varepsilon a}, \sqrt{t_2 + 3\varepsilon a}])$$

(2) If $\bar{E} \subset E$, codim $\bar{E} = n$, then

$$|\nu(A)([t, \infty]) - \nu(A \mid \bar{E})([t, \infty)| \leqslant \frac{2n}{\dim \bar{E}}$$

(3) If $E = E_1 \oplus E_2$, $A(E) = A(E_1) \oplus A(E_2)$, then

$$\nu(A) = \frac{\dim E_1}{\dim E} \nu(A \mid E_1) + \frac{\dim E_2}{\dim E} \nu(A \mid E_2)$$

(4) If $(\cdot, \cdot)'_E$ and $(\cdot, \cdot)'_F$ are other scalar products on E and F, respectively, such that $(1/\alpha) \|\cdot\|_{(\cdot)} \leqslant \|\cdot\|_{(\cdot)'} \leqslant \alpha \|\cdot\|_{(\cdot)}$, then

$$\nu(A)\left(\left[\alpha t_1, \frac{t_2}{\alpha}\right] \leqslant \nu' A)([t_1, t_2]) \leqslant \nu(A)\left(\left[\frac{t_1}{\alpha}, \alpha t_2\right]\right)$$

where in $\nu'(A)$, A^* and $|A|$ are calculated using $(\cdot, \cdot)'$ instead of $(\cdot, \cdot)$.

Corollary 1. There exists a constant C_1 such that:

(1) If $\|B\| \leqslant \varepsilon \leqslant \|A\|$, then

$$|R(A+B) - R(A)| \leqslant C_1 \sqrt{\varepsilon \|A\|} \dim E$$

(2) We have

$$|R(A \mid E_1) - R(A \mid E_2)| \leqslant C_1(\dim(E_1 + E_2) - \dim(E_1 \cap E_2)) \|A\|$$

(3) If $E = E_1 \oplus E_2$, $A(E) = A(E_1) \oplus A(E_2)$, then $R(A) = R(A \mid E_1) + R(A \mid E_2)$.

(4) If $(\cdot, \cdot)'_{E, F}$ are other scalar products on E and F respectively, such that $(1/\alpha) |\cdot|_{(\cdot)} \leqslant \|\cdot\|_{(\cdot)'} \leqslant \alpha \|\cdot\|_{(\cdot)}$, then

$$|R(A) - R'(A)| \leqslant C_1 \ln \alpha \dim E$$

where in $R'(A)$, A^* and $|A|$ are calculated using $(\cdot, \cdot)'$ instead of $(\cdot, \cdot)$.

Now we are in position to define the expansion rate of Φ. Set

$$R(\mathbf{x}, m, N) = R(d\Phi^m(\mathbf{x}) \mid V^N(\mathbf{x}))$$

$$R_n(\mathbf{x}, m, N) = R(D_n^m(\mathbf{x}) \mid V^N(\mathbf{x}))$$

$$\nu(\mathbf{x}, m, N) = \nu(d\Phi^m(\mathbf{x}) \mid V^N(\mathbf{x}))$$

$$\nu_n(\mathbf{x}, m, N) = \nu(D_n^m(\mathbf{x}) \mid V^N(\mathbf{x}))$$

By the expansion rate of Φ we mean the limit

$$R(\mathbf{x}) = \lim_{m \to \infty} \lim_{N \to \infty} \frac{1}{mN} R(\mathbf{x}, m, N)$$

The existence of this limit is proven in the next section.

4. EXISTENCE OF THE EXPANSION RATE

Theorem 1. Let μ be an S-invariant measure, then the limit

$$R(\mathbf{x}, m) = \lim_{N \to \infty} \frac{R(\mathbf{x}, m, N)}{N}$$

exists almost surely and

$$\int R(\mathbf{x}, m)\, d\mu(\mathbf{x}) = \lim_{N \to \infty} \int \frac{R(\mathbf{x}, m, N)}{N}\, d\mu(\mathbf{x})$$

If μ is S-ergodic, then $R(x, m)$ is constant almost surely.

This statement is the immediate corollary of the following result.

Lemma 1. The limit $\nu(\mathbf{x}, m) = \lim_{N \to \infty} \nu(\mathbf{x}, m, N)$ exists almost surely.

Proof. Take $\phi \in C[0, K_2'''']$. By the Proposition 1.1, given ε, we can find a large n_0 such that the inequality

$$\left| \int \phi(t)\, d\nu(\mathbf{x}, m, N)(t) - \int \phi(t)\, d\nu_n(\mathbf{x}, m, N)(t) \right| \leqslant \varepsilon$$

holds for $n \leqslant n_0$. So, it is enough to prove the existence of the limit of $\nu_n(\mathbf{x}, m, N)$ for all n. By Proposition 1.2,

$$\left| \int \phi(t)\, d\nu_n(\mathbf{x}, m, N_1 + N_2) - \int \phi(t)\, d\nu(D_n^m \mid V^{N_1 - n}(\mathbf{x}) \oplus V^{N_1 + n,\, N_1 + N_2}(\mathbf{x}))(t) \right|$$

$$\leqslant \frac{2n\, \|\phi\|}{N_1 + N_2 - 2n}$$

Let $f_N(\mathbf{x}) = N \int \phi(t)\, d\nu_n(\mathbf{x}, m, N)(t)$; then the last inequality and Proposition 1.3 imply

$$|f_{N_1 + N_2}(\mathbf{x}) - f_{N_1}(\mathbf{x}) - f_{N_2}(S^{N_1}\mathbf{x})| \leqslant \text{Const}(n)$$

so the statement of the lemma follows from the subadditive ergodic theorem applied to the sequences $f_N(\mathbf{x}) \mp \mathrm{Const}(n)$. ▌

The next step is to show that the $R(\mathbf{x}, m)$ form a subadditive sequence.

Lemma 2. $R(\mathbf{x}, m_l + m_2) \leqslant R(\mathbf{x}, m_1) + R(\Phi^{m_2}\mathbf{x}, m_2)$.

Proof. Again it is enough to replace $d\Phi^{m_i}$ by $D_n^{m_i}$ and $d\Phi^{m_1+m_2}(\mathbf{x})$ by $D_n^{m_2}(\Phi^{m_1}(\mathbf{x}))\, D_n^{m_1}(\mathbf{x})$. But in view of (5)

$$R(D_n^{m_2}(\Phi^{m_1}(\mathbf{x}))\, D_n^{m_1}(\mathbf{x}) \mid V^N(\mathbf{x})) \leqslant R_n(\mathbf{x}, m_1, N) + R_n(\Phi^{m_1}(\mathbf{x}), m_2, N+n)$$

and the lemma follows by Corollary 1.2. ▌

The application of the subadditive ergodic theorem yields the following result.

Theorem 2. If μ is also Φ-invariant, then the limit

$$R(\mathbf{x}) = \lim_{m \to \infty} \frac{R(\mathbf{x}, m)}{m}$$

exists almost surely and

$$\int R(\mathbf{x})\, d\mu(\mathbf{x}) = \lim_{m \to \infty} \int \frac{R(\mathbf{x}, m)}{m}\, d\mu(\mathbf{x})$$

If μ is (S, Φ)-ergodic, then $R(\mathbf{x})$ is constant almost surely.

Remark. Set

$$r(\mathbf{x}, c, m, N) = \ln \max_{E \subset V^N(\mathbf{x}),\ \dim E = cN} \det(d\Phi^m(\mathbf{x}) \mid E)$$

By the same subadditivity arguments it is possible to prove the existence of the limits

$$r(\mathbf{x}, c, m) = \lim_{N \to \infty} \frac{r(\mathbf{x}, c, m, N)}{N}, \qquad r(\mathbf{x}, c) = \lim_{m \to \infty} \frac{r(\mathbf{x}, c, m)}{m}$$

The calculation of $r(\mathbf{x}, c, m, N)$ can be done using the following observation. Let $A_{m,N}^{\wedge k}(\mathbf{x})$ be the kth exterior power of $d\Phi^m(\mathbf{x}) \mid V^N(\mathbf{x})$; then $(\mathbf{x}, c, m, N) = \ln \lambda_1(A_{m,N}^{\wedge cN}(\mathbf{x}))$ and therefore

$$\ln \mathrm{Tr}(A_{m,N}^{\wedge cN}(\mathbf{x})) - N \dim X \ln 2 \leqslant r(\mathbf{x}, c, m, N) \leqslant \ln \mathrm{Tr}(A_{m,N}^{\wedge cN}(\mathbf{x}))$$

so $[\ln \operatorname{Tr}(A^{\wedge cN}_{m,N}(\mathbf{x}))]/mN$ is a good approximation to $r(\mathbf{x}, c, m, N)/mN$ if m is large enough (cf. ref. 11 and the discussion in the introduction). $R(\mathbf{x})$ can be expressed in terms of $r(\mathbf{x}, c)$ as follows: $R(\mathbf{x}) = \sup_c r(\mathbf{x}, c)$.

5. RUELLE INEQUALITY

In this section we prove an infinite-dimensional counterpart of the Ruelle inequality.[7]

Theorem 3. If μ is an (S, Φ)-invariant measure, then $h(\mu) \leqslant \int R(\mathbf{x})\, d\mu(\mathbf{x})$.

Proof. This statement can be proven by exactly the same arguments as in ref. 7. Let T be a triangulation of X_0 and T_k be a k-fold barycentric subdivision of T. Since

$$\bigvee_k \bigvee_{m=-\infty}^{+\infty} \bigvee_{n=-\infty}^{+\infty} S^n \Phi^m T_k$$

is the Borel σ-algebra of $\mathbf{X}$, we have $h_{S,\Phi}(\mu) = \lim_{k\to\infty} h_{S,\Phi}(T_k, \mu)$. The last expression can be bounded by $\lim_{N\to\infty} h(T_{k,N} \mid \Phi^{-1} T_{k,N})$, where $T_{k,N} = \bigvee_{j=-N}^{N} S^j T_k$. Denote by $C_{T_{k,N}}(\mathbf{x})$ the element of $T_{k,N}$ containing $\mathbf{x}$ and let $\Gamma(k, N, \mathbf{x})$ be the number of elements $C^j_{T_{k,N}}$ of $T_{k,N}$ such that $C^j_{T_{k,N}} \cap \Phi C_{T_{k,N}}(\mathbf{x}) \neq \varnothing$; then

$$h(t_{k,N} \mid \Phi^{-1} T_{k,N}) \leqslant \int \ln \Gamma(k, N, \mathbf{x})\, d\mu(\mathbf{x})$$

Let d_k be the diameter of T_k. We can find constants n_k, α_k, β_k such that $X_{-(N+n_k), -N} \times X_{N, N+n_k}$ can be covered by $(1/\alpha_k)^{n_k}$ balls B^N_l of radius β_k such that if $\mathbf{x}'_i = \mathbf{x}''_i$ for $|i| \leqslant N$ and $(p_{-(N+n_k), -N}\mathbf{x}', p_{N, N+n_k}\mathbf{x}')$ and $(p_{-(N+n_k), -N}\mathbf{x}'', p_{N, N+n_k}\mathbf{x}'')$ belong to the same ball, then $d(p_N \Phi(\mathbf{x}'), p_N \Phi(\mathbf{x}'')) \leqslant d_k$. For k large enough the number of elements of $T_{K,n}$ such that $\Phi(B^N_l) \cap C^j_{T_{k,N}}$ is nonempty is bounded by $C^N_2(T)\, C^N_3 \exp R(\mathbf{x}, 1, N)$, where C_2 is a constant depending on the choice of the initial triangulation T, C_3 depends only on dim X, and $\mathbf{x}$ is any point in B^N_l. For such a large k we have

$$\Gamma(k, N, \mathbf{x}) \leqslant e^{R(\mathbf{x}, 1, N)} C^N_2 C^N_3 \alpha^{n_k}_k$$

Making N go to infinity gives

$$h_{S,\Phi}(T_k, \mu) \leqslant \ln C_2 + \ln C_3 + \int R(\mathbf{x}, 1)\, d\mu(\mathbf{x})$$

and therefore

$$h_{S,\Phi}(\mu) \leqslant \ln C_2 + \ln C_3 + \int R(\mathbf{x}, 1)\, d\mu(\mathbf{x})$$

Replacing Φ by Φ^m, we obtain

$$mh_{S,\Phi}(\mu) \leqslant \ln C_2 + \ln_3 + \int R(\mathbf{x}, m)\, d\mu(\mathbf{x})$$

Dividing by m and passing to the limit $m \to \infty$ provides the statement claimed. ▮

6. SBR MEASURE

In ref. 5 a measure for Φ with the properties similar to those of the SBR measure in the finite-dimensional case was constructed under the assumption that Φ is a small perturbation of a system of noninteracting hyperbolic mappings.

Here we recall this construction. Let Λ be a hyperbolic attractor for f such that $f|_\Lambda$ is topologically transitive. Then the tangent space at every point $x \in \Lambda$ can be decomposed into the sum $T_x X = E^{(u)}(x) + E^{(s)}(x)$, where

$$df^n|_{E^{(u)}(x)} \geqslant K_3 \kappa_3^n \tag{6}$$

$$df^{-n}|_{E^{(s)}(x)} \geqslant K_3 \kappa_3^n \tag{7}$$

and the angle

$$\angle(E^{(u)}(x), E^{(s)}(x)) \geqslant \gamma_3 \tag{8}$$

for some constants K_3, $\kappa_3 > 1$, and γ_3.

Consider a sequence of embeddings I_N: $X_N \to \mathbf{X}$ such that $\|I_N\|_{C^2} \leqslant K_4$ and $P_N I_N = \text{id}$ and let $\Phi_N = P_N \circ \Phi \circ I_N$ and $f_N = P_N \circ \mathbf{f} \circ I_N$ so that $(f_N(x^{(N)}))_i = f(x_i^{(N)})$. The following statements hold if K_1 and κ_1 are small enough and J is close to identity in C^2-norm[5]:

(1) Φ_N has an attractor Λ_N on which Φ_N is conjugated to

$$f_N|_{\prod_{i=-N}^{N} \Lambda}$$

and Φ has an attractor $\mathbf{\Lambda}$ on which Φ is conjugated to

$$\mathbf{f}|_{\prod_{\mathbb{Z}} \Lambda}$$

In both cases the conjugation is close to identity;

(2) Φ_N is hyperbolic on Λ_N and Φ is hyperbolic on $\mathbf{\Lambda}$. Moreover, the constants K_3, κ_3, and γ_3 can be chosen so that formulas (6)–(8) hold with f replaced by Φ_N or Φ.

(3) Let Π_f be a Markov partition for f; Π_{f_n} and $\Pi_{\mathfrak{f}}$ be partitions whose elements are products of elements of Π and Π_{Φ_N}, and Π_Φ be their images under the above-mentioned conjugations. Π allows us to identify $f|_\Lambda$ with a subshift of the finite type (Σ_A, σ) with an alphabet $\{1 \cdots l\}$, where $l = \text{Card}(\Pi)$. Then Φ_N and Φ are semiconjugated to subshifts $(\Sigma^{(N)}_{A_N}, \sigma)$ and $(\mathbf{\Sigma}_{\mathbf{A}}, \sigma)$, respectively, with alphabets $\{1 \cdots l\}^{2N+1}$ and $\{1 \cdots l\}^{Z}$, respectively, and the transition matrices given by

$$A_N(\vec{\imath}^{(1)}, \vec{\imath}^{(2)}) = \prod_{j=-N}^{N} A_{i_j^{(1)}, i_j^{(2)}}, \qquad \mathbf{A}(\vec{\imath}^{(1)}, \vec{\imath}^{(2)}) = \prod_{j \in Z} A_{i_j^{(1)}, i_j^{(2)}}$$

The mapping S acts on $\mathbf{\Sigma}$ as the space shift $S(\vec{\imath})^j_k = i^j_{k+1}$, where the superscript signifies the time coordinate and the subscript stands for the space coordinate. Let μ_N be the SBR measure on Λ_N and $\hat{\mu}_N$ be its pullback to Σ_N. It is proven in ref. 5 that the $\{\mu_N\}$ converge to a measure μ on $\mathbf{\Lambda}$, that is, if $g: \mathbf{X} \to R$ is a function depending only on a finite number of coordinates, then

$$\int g \, d\mu_n \to \int g \, d\mu \tag{9}$$

Moreover, if $\hat{\mu}$ is the pullback of μ on $\mathbf{\Sigma}$ then the conditional expectations converge as well:

$$\begin{aligned} &\hat{\mu}_N(i^0_0 \mid i^0_{-1} \cdots i^0_{-N}, i^{-1}_N \cdots i^{-1}_{-N} \cdots i^{-j}_k \cdots) \\ &\quad \to \hat{\mu}(i^0_0 \mid i^0_{-1} \cdots i^0_{-L} \cdots \vec{\imath}^{-1} \vec{\imath}^{-2} \cdots \vec{\imath}^{-j} \cdots) \end{aligned}$$

The measure μ is mixing with respect to both S and Φ.

We now specify I_N to be the periodic embedding $(I_N(x^{(N)}))_i = x^{(N)}_j$, where $i \equiv j \bmod(2N+1)$. The immediate corollary of the above-mentioned properties of λ and S-invariance of λ_N is the following statement.

Corollary 2. $h_{S,\Phi}(\mu) = \lim_{N \to \infty} (1/N)\, h_{\Phi_N}(\mu_N)$.

7. PESIN FORMULA

Let μ be SBR measure for Φ. Since μ is both S and Φ-ergodic, $R(m) = R(m, \mathbf{x})$ does not depend on $\mathbf{x}$. We write $R_\mu(\Phi) = \lim_{m \to \infty} [R(m)/m]$.

Theorem 4. $h_{S,\Phi}(\mu) = R_\mu(\Phi)$.

Proof. We combine Corollary 2 with Pesin formula for Φ_N to get

$$h_{S,\Phi}(\mu) = \lim_{N\to\infty} \lim_{m\to\infty} \frac{1}{mN} \int R(d\Phi_N^m)\, d\mu_N(x^{(N)})$$

Our concern now is to show that the last two limits can be interchanged. Consider the decomposition

$$T_{x^{(N)}}X_N = E_N^{(u)}(x^{(N)}) + E_N^{(s)}(x^{(N)})$$

and let a new metric $(\cdot,\cdot)'$ on $T_{x^{(N)}}X_N$ be given by the conditions $\|v\|_2' = \|v\|_2$ for $v \in E_N^{(u)}(x^{(N)})$ or $v \in E_N^{(s)}(x^{(N)})$ and $E_N^{(u)} \perp' E_N^{(s)}$. Since

$$\frac{\|\cdot\|_2}{\sqrt{2}} \leqslant \|\cdot\|_2' \leqslant \frac{\|\cdot\|_2}{\sqrt{1-\cos\gamma_3}}$$

Corollary 1.4 implies

$$|R(d\Phi_N^m) - \ln\det(d\Phi_N^m \,|\, E_N^{(u)}(x^{(N)}))| \leqslant C_4 N$$

that is,

$$\left|\frac{1}{mN} R(d\Phi_N^m) - \frac{1}{mN}\ln\det(d\Phi_N^m \,|\, E_N^{(u)}(X^N)\right| \leqslant \frac{C_4}{m}$$

But since

$$\begin{aligned}&\det(d\Phi_N^{m_1+m_2} \,|\, E_N^{(u)}(x^{(N)}))\\ &\quad = \det(d\Phi_N^{m_2} \,|\, E_N^{(u)}(\Phi_N^{m_1}(x^{(N)})))\det(d\Phi_N^{m_1} \,|\, E_N^{(u)}(x^{(N)}))\end{aligned}$$

the last inequality gives

$$\left|\lim_{m\to\infty} \frac{1}{mN}\int R(d\Phi_N^m)\, d\mu_N - \frac{1}{m_0 N}\int R(d\Phi_N^{m_0})\, d\mu_N\right| \leqslant \frac{C_4}{m_0}$$

and hence

$$h_{S,\Phi}(\mu) = \lim_{m\to\infty} \lim_{N\to\infty} \frac{1}{mN}\int R(d\Phi_N^m(x^{(N)})\, d\mu_N$$

as claimed. Let us calculate the interior limit. By the same arguments as in the proof of Theorem 1, given ε, we can find L_0 so large that for $L \geqslant L_0$

$$\left| R(d\Phi'''_N) - \sum_{j=-N/L}^{N/L} R(d\Phi'''_N \mid V^{jL+1,(j+1)L}(x^{(N)})) \right| \leqslant \varepsilon N$$

and, since μ_N is S-invariant,

$$\left| \frac{1}{N} \int R(d\Phi'''_N)\, d\mu_N(x^{(N)}) - \frac{1}{L} \int R(d\Phi'''_N \mid V^L(x^{(N)}))\, d\mu_N(x^{(N)}) \right| \leqslant \varepsilon$$

In other words,

$$h_{S,\Phi}(\mu) = \lim_{m\to\infty} \lim_{L\to\infty} \lim_{N\to\infty} \frac{1}{mL} \int R(d\Phi'''_N \mid V^L(x^{(N)}))\, d\mu_N$$

By the short-range conditions (1)–(2) there is $C_5 = C_5(m, L)$ such that

$$\|(d\Phi'''_N(x^{(N)}) \mid V^L(x^{(N)})) - (d(\Phi''' \circ I_N)(x^{(N)}) \mid V^L(x^{(N)}))\| \leqslant C_5 \kappa_1^N$$

Therefore

$$h_{S,\Phi}(\mu) = \lim_{m\to\infty} \lim_{L\to\infty} \lim_{N\to\infty} \frac{1}{mL} \int R(d\Phi''' \mid V^L(I_N x^{(N)}))\, d\mu_N(x^{(N)})$$

By construction of the measure μ [see formula (9)] the interior limit is equal to $\int R(d\Phi''' \mid V^L(\mathbf{x}))\, d\mu(\mathbf{x})$ and hence

$$h_{S,\Phi}(\mu) = \lim_{m\to\infty} \lim_{l\to\infty} \frac{1}{mL} \int R(d\Phi''' \mid V^L(\mathbf{x}))\, d\mu(\mathbf{x})$$

$$= \lim_{m\to\infty} \frac{R(m)}{m} = R_\mu(\Phi). \quad \blacksquare$$

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